Bonds 101

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The following is a high level overview of bonds, (including pricing, duration and the impact of maturity, yield and coupon rates on duration and price) which hopefully provides a thorough and not too painful explanation of how bond prices are impacted by changes in the market and just what duration measures.

Most bonds sold in public markets pay interest twice a year, (these are called coupons) and have a face value of $1,000.00. The face value means that at the end of the life of the bond, the owner will receive $1,000.00. The interest rate quoted with the bond is an annual rate, thus a 5 year, 6% bond would pay (1,000 x 6% / 2) = $30 twice a year.

Zero coupon bonds pay no annual interest (or coupons). Thus a zero coupon bond will simply pay the owner $1,000 at the end of the life of the bond. The interest is in effect paid in a lump sum when the owner buys the bond for less than $1,000. i.e. buying a $1,000 bond with one year to maturity for $900, gives the owner an 11% return -> ($1000 - $900) / $900 = 11%.

First let's price a bond. We'll use a simple example of a five year bond that pays 4% interest only once a year, (remember most bonds pay twice a year). Let's assume that the current yield for this bond is 3%. What is the price? The price is simply the present value of all the future cash flows, namely $40 every year until year 5 when the owner receives the $1,000 face value in addition to the $40 coupon.

Present value = (Cash Received at time t) / $(1 + \text{yield})^t$

The coupons are $40 ($1,000 face value x 4%)

So the present value of the first year's coupon of $40 is ($40 / $(1+ 0.03)^1$ = $38.83

The present value of the second year's coupon of $40 is ($40 / $(1+0.03)^2$ = $37.70

Notice that as the payments get further away from today, they decline in value. You may also hear of this process being referred to as the discounted cash flow. The 3% yield is also the discount rate in this example.

For all five years, we have the following.

Year	Cash Flow	PV of Cash Flow
1	40.00	38.83
2	40.00	37.70
3	40.00	36.61
4	40.00	35.54
5	1,040.00	897.11
TOTAL (Price)		**$1,045.80**

Thus at a 3% yield, the price for a 5 year, 4% bond with one payment a year is $1,045.80. Notice that a bond that pays 4% costs MORE than its face value of $1,000 when the yield is 3%, lower than the rate the bond pays. Bonds priced above face value are said to be priced at a premium.

What if the yield were 5%, higher than the rate on the bond, how would this affect price?

Year	Cash Flow	PV of Cash Flow
1	40.00	38.10
2	40.00	36.28
3	40.00	34.55
4	40.00	32.91
5	1,040.00	814.87
TOTAL (Price)		**$956.71**

When yields are higher than the coupon rate the bond pays, the bond price is below the face value of $1,000. Bonds priced below the face value are said to be priced at a discount. Why is this?

In order to get a 5% yield, meaning for an investor to get a 5% return on a bond that is only paying 4%, the price of the bond needs to be LESS than the face value of $1,000. For an investor to receive a 3% return on a bond that pays 4%, the investor would have to pay MORE than the face value of $1,000.

Please also note that bonds are priced using a basis of 100, thus a bond with a face value of $1,000 that is priced at $956.71 would be referred to at a price of 95.671. In this way bonds with face values of $1,000, $5,000 or $10,000 can be easily compared, by simply dividing their respective prices by either 10, 50 or 100 to get them all to an equal base of 100. For example a bond with a face value of $5,000 that is priced at $4,890 would be quoted at $4,890/50 = 97.80. A bond with a face value of $10,000 and priced at $11,310 would be quoted at $11,310 / 100 = 113.10. This makes it easier to compare the $5,000 bond to the $10,000 bond.

But what about our original question concerning inflation? Interest rates reflect the cost of borrowing money. That cost is affected by the rate of inflation. When inflation is low, interest rates tend to be lower than when inflation is high. In inflationary periods, the purchasing power of the original amount loaned declines significantly, thus the lender must increase the interest rate charged to make up for that loss of purchasing power. For example, with no inflation and a lender charges you 5% to borrow $100 for one year. If inflation is known to be 2% in the next year, the lender would charge you at least 7% (5% cost of borrowing + 2% inflation). Thus in inflationary periods, interest rates are higher.

Now that we know how to price a bond and that interest rates are affected by inflation, how do we estimate what will happen to the prices of bonds when interest rates change? We do that using **Duration** and **Convexity**.

Bond duration can be thought of as the *weighted average term to maturity of a bond's cash flows*. This means that duration is a measure of the sensitivity of the price of a bond to a change in interest rates. For those familiar with Capital Asset Pricing Model (CAPM), duration is analogous to the way Beta is a measure of stockǿs sensitivity to fluctuations in the market. Thus duration allows us to evaluate the exposure to changes in interest rates of bonds with differing maturities and coupon payments by taking into account both interim and final coupon payments and weighting earlier cash flows as being more important that latter ones.

Letǿs now take our first example and convert it into a more typical bond which pays coupons on a semi-annual basis. To do this we will use periods (meaning a six month period) rather than years and will cut the coupon payment in half, from $40 to $20. The yield of 3% will also be cut in half to reflect semi-annual payments, rather than annual. This gives us the following Present Value (PV) using semi-annual coupon payments.

Period	Cash Flow	PV of Cash Flow
1	20.00	$20 / (1.03/2)^1 = 19.70$
2	20.00	$20 / (1.03/2)^2 = 19.41$
3	20.00	$20 / (1.03/2)^3 = 19.13$
4	20.00	18.84
5	20.00	18.57
6	20.00	18.29
7	20.00	18.02
8	20.00	17.75
9	20.00	17.49
10	1,020.00	878.90
TOTAL (Price)		**$1,046.11**

Since duration is the weighted average term to maturity of a bondǿs cash flow, then we need to weight the present value of each payment.

Period	Cash Flow	PV of Cash Flow	% of Toal PV	% x Period
1	20.00	19.70	19.70 / 1,046.11 = 1.88%	1.88% * 1 = 0.02
2	20.00	19.41	19.41 / 1,046.11 = 1.86%	1.86% * 2 = 0.04
3	20.00	19.13	19.13 / 1,046.11 = 1.83%	1.83% * 3 = 0.05
4	20.00	18.84	1.80%	0.07
5	20.00	18.57	1.77%	0.09
6	20.00	18.29	1.75%	0.10
7	20.00	18.02	1.72%	0.12
8	20.00	17.75	1.70%	0.14
9	20.00	17.49	1.67%	0.15
10	1,020.00	878.90	84.02%	8.40
TOTAL		**$1,046.11**	**100.00%**	**9.18**

Now we have a Duration of 9.18, but that is in six month periods, so to annualize this duration, divide by 2 to get an annualized duration of 4.59 = (9.18 / 2).

Now what? We have duration of 4.59 but how do we use it? The way to use this measure is to calculate the modified duration or volatility.

Volatility (percent) = Duration / (1+y) where y is the yield to maturity prior to changes in interest rates. In our example above, the yield is 3%, thus the volatility of this bond is as follows:

Volatility (Modified Duration) = 9.18 / (1+ 0.03) = 8.91 and to annualize = 8.91 / 2 = 4.46.

This means that for every 1% change in yields, the price of the bond will change 4.46%. So a 1% increase in yields will cause the price of the bond to decrease by 4.46%, remember that yield and price are inversely related.

So what does this tell us about the price sensitivity of bonds? The price of a bond with a higher duration will fluctuate more with changes in interest rates than a bond with a lower duration. We can also see that duration is positively related to the bondꞌs maturity. *When all else is held constant, the longer the term to maturity, the higher the bond's duration.*

Letꞌs look at three bonds, with maturities of 1 year, 5 years and 10 years, all with an 8% coupon and 12% yield. For the 1 year bond, the calculation looks like this.

Period	Cash Flow	PV of Cash Flow	% of Total PV	% x Period
1	40.00	37.74	3.92%	0.04
2	1,040.00	925.60	96.08%	1.92
TOTAL		**963.33**	**100.00%**	**1.96**

We have duration of 1.96, which when annualized is 1.96 / 2 = 0.98.

To get the modified duration or volatility,

Duration / (1+y) => 1.96 / (1+0.12/2) = 1.85 / 2 (to annualize) = 0.92

By performing the same calculations for the remaining bonds we get the following chart.

Maturity (in years)	Duration (in years)	Volatility	Price (PV of Cash Flow)
1	0.9804	0.92	963.33
5	4.14	3.91	852.80
10	6.61	6.23	770.60

Notice that duration above is consistently less than maturity and the price is below the face value of $1,000. As the bondôs maturity increases, from 1 year to 5 years, to 10 years, the volatility to changes in interest rates increases from 0.92% change for every 1% change in interest rates with the 1 year bond to a 6.23% change for every 1% change for the 10 year bond.

What if for the same three bonds, the coupon rate is greater than the yield? Letôs keep the coupon at 8% and make the yield 6%.

Maturity (in years)	Duration (in years)	Volatility	Price (PV of Cash Flow)
1	0.9809	0.95	1,019.13
5	4.25	4.13	1,085.30
10	7.29	7.07	1,148.77

Again, duration is still less than maturity, but is greater when yield is less than the coupon rate! Duration is consistently higher when the yield is 6% versus 12% for the same 8% bond.

The prices on these bonds are all greater than face value of $1,000 when the yield is below the bondôs coupon rate. Notice in this case as in the 12% yield, as the bondôs maturity increases, from 1 year to 5 years, to 10 years, the volatility to changes in interest rates increases.

Regardless of whether the yield is greater than or less than the bond's coupon rate, an increase in maturity increases volatility with respect to changes in interest rates. This can be thought of as the whip effect, where small movement at the handle results in large changes out at the tail.

Weôve looked at the impact of maturity on a bondôs sensitivity to changes in interest rates, now letôs look at the impact of coupon rates. Letôs keep the bondôs maturity at 5 years, and look at coupon rates of 4%, 6%, 8% and 10% with a yield of 5%.

Coupon Rate	Duration (in years)	Volatility	Price (PV of Cash Flow)
4%	4.57	4.46	956.24
6%	4.41	4.30	1,043.76
8%	4.27	4.17	1,131.28
10%	4.16	4.05	1,218.80

As the coupon rate increases, the bond's duration/volatility falls.

This is because with a higher coupon rate, more cash is received earlier and accounts for a greater percentage of the present value of the bond.

For the 4% bond from above, the first 3 years of coupon payments account for 11.52% of the total present value of the bond. (Add the % of Total PV for periods 1-6 together to get 11.52%)

Period	Cash Flow	PV of Cash Flow	% of Total PV	% x Period
1	20.00	19.51	2.04%	0.02
2	20.00	19.04	1.99%	0.04
3	20.00	18.57	1.94%	0.06
4	20.00	18.12	1.89%	0.08
5	20.00	17.68	1.85%	0.09
6	20.00	17.25	1.80%	0.11
7	20.00	16.83	1.76%	0.12
8	20.00	16.41	1.72%	0.14
9	20.00	16.01	1.67%	0.15
10	1,020.00	796.82	83.33%	8.33
TOTAL		**956.24**	**100.00%**	**9.14/2 = 4.57**

For the 10% bond from above, the first 3 years of coupon payments account for 22.60% of the total present value of the bond.

Period	Cash Flow	PV of Cash Flow	% of Total PV	% x Period
1	50.00	48.78	4.00%	0.04
2	50.00	47.59	3.90%	0.08
3	50.00	46.43	3.81%	0.11
4	50.00	45.30	3.72%	0.15
5	50.00	44.19	3.63%	0.18
6	50.00	43.11	3.54%	0.21
7	50.00	42.06	3.45%	0.24
8	50.00	41.04	3.37%	0.27
9	50.00	40.04	3.28%	0.30
10	1,050.00	820.26	67.30%	6.73
TOTAL		**1,218.80**	**100.00%**	**8.31/2 = 4.16**

For a given 5 year maturity period and an annual yield of 5%, *the bond with the higher coupon is also the one with the shortest duration, thus least sensitivity to changes in interest rates.*

Finally, what about zero coupon bonds? These are bonds that pay no coupons during their lifetime. Thus the duration is exactly the same as their maturity. Letôs see why.

Letôs look at a 3 year zero coupon bond with a yield of 5%.

Period	Cash Flow	PV of Cash Flow	% of Total PV	% x Period
1	0.00	0.00	0.00%	0.00
2	0.00	0.00	0.00%	0.00
3	0.00	0.00	0.00%	0.00
4	0.00	0.00	0.00%	0.00
5	0.00	0.00	0.00%	0.00
6	1,000.00	862.30	100.00%	6.00
TOTAL		**862.30**	**100.00%**	**6.00/2 = 3**

100% of the value of the bond comes from the return of the face value at the end of the life of the bond. So a zero coupon bond, regardless of the yield has a duration equal to its maturity. Its modified duration or volatility is different though based on the yield. In this example, the volatility would be 3/(1+0.05) = 2.86

Letҳ look at zero coupon bonds with maturities of 1 year, 3 years, 5 years and 10 years at 5% yield.

Maturity (in years)	Duration (in years)	Volatility	Price (PV of Cash Flow)
1	1	0.98	951.81
3	3	2.93	862.30
5	5	4.88	781.20
10	10	9.76	610.27

As maturity increases, so does sensitivity to changes in interest rates.

Finally, letҳ look at how yield changes the sensitivity of a bondҳ price to changes in interest rates. *All other factors being held constant, the higher the bond's yield to maturity, the lower its duration, thus sensitivity to changes in interest rates.*

Letҳ take a 10 year bond with an 8% coupon and yields of 5%, 10% and 15%.

Yield	Duration (in years)	Volatility	Price (PV of Cash Flow)
5%	7.39	7.21	1,233.84
10%	6.84	6.51	875.38
15%	6.25	5.82	643.19

As the yield increases, volatility (sensitivity to changes in interest rates) decreases.

Now that we know how the various factors affect duration, how do we use duration to aid in investment decisions? In general, when interest rates are expected to rise, investors move towards bonds with lower duration.

Warning, the next part on convexity is only for those who just canӴ get enough of equations.

Convexity

So what is convexity? For smaller changes in yield (meaning smaller changes in interest rates) duration does a good job in estimating the actual price. This is a straight linear equation though, and price does not change in a perfectly linear fashion when yields change, thus duration is less reliable when we are looking at larger changes in yields. Convexity can be thought of as the degree of curvature that exists in the price to yield relationship or for those who are more mathematically inclined, the second derivative of duration.

To calculate convexity, letҳ use our 5 year, 8% bond at a 15% yield.

Period	Cash Flow	PV of Cash Flow	PV * t * (t + 1)
1	40.00	37.21	37.21 * 1 * (1 + 1) = 74.42
2	40.00	34.61	34.61 * 2 * (2 + 1) = 207.68
3	40.00	32.20	32.20 * 3 * (3 + 1) = 386.38
4	40.00	29.95	599.04
5	40.00	27.86	835.87
6	40.00	25.92	1088.58
7	40.00	24.11	1350.17
8	40.00	22.43	1614.82
9	40.00	20.86	1877.70
10	1,040.00	504.60	55,506.19
TOTAL		**759.76**	**63,540.84**

Convexity is the Sum of the $\frac{(PV * t * (t+1)) / \text{Price of the bond}}{(1+ \text{yield})^2} =$

$\frac{64,540.84 / 759.76}{(1+ 0.15/2)^2} = \frac{83.63}{(1.075)^2} = \frac{83.63}{1.156} = 72.34$

To annualize, since this is a second derivative of duration, we take $72.34 / 2^2 = 18.1$

Now we can calculate the percentage of the price change not explained by duration as

$0.5 * \text{Convexity} * (\text{Yield Change})^2 * 100$

Thus for a 4% increase in yields, the percentage of the price change not explained by duration is

$0.5 * 18.1 * 0.04^2 * 100 = 1.45\%$